PMM U.S.S.R., Vol.44, pp.51-59
0021-8928/81/1 0051 $7.50/0
UDC 534.2:532

ON NONLINEAR ACOUSTICS APPROXIMATION IN PROBLEMS OF GAS OSCILLATIONS IN PIPES *

A. N. KRAIKO and A. L. NI

One-dimensional nonlinear oscillations of perfect gas in pipes are considered. The dependence of the slope of characteristics on parameter perturbations and the possibility of weak shocks are taken into account, but the variation of entropy and Riemann invariants in them are neglected. Particular attention is given to cases in which it is possible to disregard the interaction between waves of various sets. Near-resonance oscillations for which nonlinear effects and formation of shocks are particularly important are analyzed as an example.

Among the publications that are relevant to further investigations of nonlinear acoustics of perfect gas and other media we would point out /1—19/. The most important of these for the derivation of concrete results is /1/, where a comparatively simple, but formal method is given for constructing a class of discontinuous solutions, and its effectiveness is illustrated. The method used here is closer to the physical approach developed in /2—9/ and, unlike /1/ is based on fairly clear considerations related to characteristics, their intersection, etc.

The statement and solution of problems of nonlinear periodic oscillations are not possible (or unsubstantiated) without the clarification of a number of fundamental aspects. Thus, physically obvious and experimentally observable periodic modes with shocks cannot in the main be defined by the exact equations of perfect gas. For instance, the entropy increase allowed for in the "exact formulation" moves the system, in the absence of a mean flow through the pipe, unavoidably away from the resonance mode, as was observed in /20/. Hence the analysis of near-resonance modes is only possible using the approximate model that takes into account heat removal through the walls, or simplified equations in which entropy increase in the shock is neglected. The use of simplified equations is also justified because in such problems oscillations, owing to the formation of shocks, are small away from and close to resonance (although they are greater in the second case). This circumstance, although impeding the use of numerical methods that are effective in other gasdynamic problems, guarantees the accuracy of approximate equations.

On the basis of the above considerations the authors have developed a method which uses simplified equations and a special numerical procedure, and has several advantages (particularly as regards simplicity and the range of solvable problems) over those proposed in the cited publications. The inclusion in the numerical procedure of a natural algorithm for the determination of shocks generated along the pipe length eliminates the problem of merging smooth sections of solutions which is possibly the weakest link in almost all investigations carried out so far.

1. Let us consider one-dimensional oscillations in a pipe at low velocities and nearly homogeneous remaining parameters which will be denoted by subscript zero. We define velocity u and the speed of sound a by $u = a_0 \varepsilon u'$ and $a = a_0(1 + \varepsilon a')$, respectively, where ε is the deviation of u and a from $u_0 = 0$ and a_0 and is selected so that $\max(|u'|, |a'|) = 1$. Parameter ε does not necessarily coincide with the amplitude of external actions that may be specified at the left-hand $(x = 0)$ or right-hand $(x = X)$ ends of the pipe. Shocks may arise in the pipe, whose amplitude does not exceed 2ε and the entropy increase in each shock is equal $O(\varepsilon^3)$. As stated in the introduction, we disregard this increase, and assume that the gas entropy does not deviate from its mean value. On these assumptions the flow at every point is completely defined by u and a or by their functions, the Riemann invariants $J^\pm$. For the perfect gas $J^\pm = u \pm 2a / (\varkappa - 1)$, where $\varkappa$ is the adiabatic exponent.

The continuity regions $J^+ (J^-)$ remain constant along the $c^+(c^-)$-characteristics, while at intersections of shocks of "opposite" sets by characteristics they and the entropy do not change by more than $O(\varepsilon^3)$. If a characteristic intersects in the pipe N shocks, the over-all change of $J^\pm$ does not exceed $O(\varepsilon^3 N)$, and for $\varepsilon^3 N \ll \varepsilon$ it is negligibly small in comparison with the deviation of invariants from their "mean" values. For $N \ll \varepsilon^{-2}$ even in the presence of weak shocks we have, with an accuracy to ε, $J^+ = J^+[\xi(t, x)]$ and $J^- = J^-[\eta(t, x)]$, where ξ and η are characteristic variables that remain constant along the c^+- and c^--characteristics. The nonlinearity effects have then a cumulative effect, giving rise (owing to the dependence of velocity of characteristics on parameters) to intersections with each other and with shocks of the same family.

*Prikl.Matem.Mekhan.,44,No.1,77-88,1980

Besides u' and a' we introduce x', t' and $J^{\pm\prime}$ by the equalities

$$x = x'L, \quad t = t'L / a_0, \quad J^{\pm} = a_0 \varepsilon J^{\pm\prime} \pm 2a_0 / (\varkappa - 1) \tag{1.1}$$

where L is a characteristic dimension and t is the time. Denoting by T the characteristic time, for instance, the period of the periodic flow, we have by virtue of (1.1) $T = T'L / a_0$, $X = X'L$ and $J^{\pm\prime} = u' \pm 2a' / (\varkappa - 1)$, and in accordance with the choice of ε, $|J^{\pm\prime}| \sim O(1)$. Below we shall use only "primed" quantities with the primes omitted. Boundary conditions for invariants $J^+(t, 0) = F^+(t)$ and $J^-(t, X) = F^-(t)$ can be formulated with continuous or discontinuous $F^{\pm}(t) \sim O(1)$ which we assume such that at any instants t_1 and t_2

$$\left| \int_{t_1}^{t_2} F^{\pm}(t)\, dt \right| \leqslant A^{\pm} T \tag{1.2}$$

with constants $A^{\pm} \sim O(1)$.

In the used here notation the equations of characteristics are of the form

$$\begin{aligned} \left(\frac{\partial x}{\partial t}\right)_{\xi} &= 1 + \varepsilon (u + a) = 1 + \varepsilon \left[\frac{\varkappa + 1}{4} J^+(\xi) + \frac{3 - \varkappa}{4} J^-(\eta)\right] \\ \left(\frac{\partial x}{\partial t}\right)_{\eta} &= -1 + \varepsilon (u - a) = -1 + \varepsilon \left[\frac{\varkappa + 1}{4} J^-(\eta) + \frac{3 - \varkappa}{4} J^+(\xi)\right] \end{aligned} \tag{1.3}$$

Taking into account that $\xi\,(\eta)$ is constant along every $c^+\,(c^-)$-characteristic, taking for $\xi\,(\eta)$ the instants at which the $c^+\,(c^-)$-characteristic leaves the cross section $x = 0\ (X)$, and integrating the obtained equations, we obtain

$$\begin{aligned} c^+:\quad t - x &= \xi - \varepsilon \left[\frac{\varkappa + 1}{4} J^+(\xi)(t - \xi) + \frac{3 - \varkappa}{4} \int_{\xi}^{t} J^-(\eta)\, dt\right] \\ c^-:\quad t + x &= \eta + X + \varepsilon \left[\frac{\varkappa + 1}{4} J^-(\eta)(t - \eta) + \frac{3 - \varkappa}{4} \int_{\eta}^{t} J^+(\xi)\, dt\right] \end{aligned} \tag{1.4}$$

The integral in the first equation is calculated with $\xi = \text{const}$ and in the second, with $\eta = \text{const}$. Shock waves are generated as the consequence of intersection of characteristics of the same set. Since the velocity of a weak shock is, to within ε, equal to the half-sum of velocities of characteristics converging on it /17/, hence, for instance, for a shock propagating to the right

$$\frac{dx_s}{dt} = 1 + \varepsilon \left[\frac{\varkappa + 1}{8} J^+(\xi_{s1}) + \frac{\varkappa + 1}{8} J^+(\xi_{s2}) + \frac{3 - \varkappa}{4} J^-(\eta_s)\right] \tag{1.5}$$

where it is taken into account that J^- at an intersection of a weak shock of the "first set" (the "c^+-shock") does not change, and the subscript s denotes parameters at the shock while subscripts 1 and 2 denote parameters on the opposite sides of it. By virtue of (1.4) $t - \xi = x + O(\varepsilon X)$ and $t - \eta = X - x + O(\varepsilon X)$. Hence it is possible to substitute for (1.4) the formulas

$$\begin{aligned} c^+:\quad t - x &= \xi - \varepsilon \left[\frac{\varkappa + 1}{4} J^+(\xi) x + \frac{3 - \varkappa}{4} \int_{\xi}^{t} J^-(\eta)\, dt\right] + O(\varepsilon^2 X) \\ c^-:\quad t + x &= \eta + X + \varepsilon \left[\frac{\varkappa + 1}{4} J^-(\eta)(X - x) + \frac{3 - \varkappa}{4} \int_{\eta}^{t} J^+(\xi)\, dt\right] + O(\varepsilon^2 X) \end{aligned} \tag{1.6}$$

Equations (1.5) and (1.6) with the initial distributions of $J^{\pm}$ and boundary conditions, for instance, for J^+ at the left and for J^- at the right-hand pipe ends, completely determine the flow. Unlike in the linear (acoustic) approximation, the initial distributions are "forgotten" at $t \gg X$, and the periodic boundary conditions "produce" solutions periodic in t. The analysis is considerably simplified when the interaction between waves propagating in different directions is negligible, which makes it possible to neglect in the right-hand sides of (1.5) and (1.6) terms containing factors $(3 - \varkappa)$. The c^+- and c^--characteristics are then represented by the straight lines

$$\begin{aligned} c^+:&\quad t - x = \xi - \varepsilon (\varkappa + 1) J^+(\xi) x / 4 \\ c^-:&\quad t + x = \eta + X + \varepsilon (\varkappa + 1) J^-(\eta)(X - x) / 4 \end{aligned} \tag{1.7}$$

and the shock equations, after rejection of terms of order $\varepsilon^2 T$, become

$$c^{\pm}:\quad \frac{dt_s}{dx} = \pm 1 - \varepsilon \frac{\varkappa + 1}{8} (J_{s1}^{\pm} + J_{s2}^{\pm}) \tag{1.8}$$

$$(J_{si}^{+} = J^+(\xi_{si}), \quad J_{si}^{-} = J^-(\eta_{si}))$$

The flows defined by each pair of Eqs. (1.7) and (1.8) represent, as in /17—19/, sequencies of simple waves separated by weak shocks.

In the problems considered here the disregard of interaction between waves of different sets was used and partly substantiated in /5—8/. It should be, however, stressed that this approximation is far from being always justified in problems of this type. The disregard of terms of order εT in (1.7) and (1.8) is admissible, besides in the obvious case of $\varkappa = 3$, first, for long pipes for which $n \equiv X / T \gg 1$ and, second, when $n \sim 1$ for a part of near-resonance modes. Note that when the latter is admissible, it is possible to substitute for the integration of (1.8) the following method of replacing the many-valued solution by a single-valued discontinuous solution. Let us consider, for instance the c^+-wave for which the function $t = t(J^+)$ at cross section $x = \text{const}$ determined by (1.7) proves to be many-valued. Then the shock which eliminates the many-valuedness intersects that cross section at instant $t = t_s(x)$ which is determined by the condition

$$\int_{J_{s1}^+(x)}^{J_{s2}^+(x)} [t(J^+) - t_s]\, dJ^+ = 0$$

In this case the rule is derived in practice as in /17/. With the use of (1.7) and (1.8) it can be written in the form

$$\int_{\xi_{s1}}^{\xi_{s2}} J^+(\xi)\, d\xi = \frac{1}{2}(J_{s1}^+ + J_{s2}^+)(\xi_{s2} - \xi_{s1}) \tag{1.9}$$

Formula $t_s = t(\xi_{s1}) = t(\xi_{s2}) = [t(\xi_{s1}) + t(\xi_{s2})] / 2$ with function $t(\xi)$ determined by (1.7) was used for t_s when deriving (1.9). It can be shown that (1.9) is also valid when the shock wave is already present at $x = 0$, i.e. that the distribution of $J^+(\xi)$ is discontinous.

To complete the exposition of general considerations we formulate an integral law which may, for example, be used for the control of calculations. This law of "conservation of the invariant" is analogous to the law of conservation of momentum in a simple wave, and can be written in the form

$$\int_{t_1}^{t_2} J^+ dt - \frac{1}{2}(J_1^+ + J_2^+)(t_2 - t_1) = \int_{\xi_1}^{\xi_2} J^+ d\xi - \frac{1}{2}(J_1^+ + J_2^+)(\xi_2 - \xi_1) \tag{1.10}$$

For c^--waves equalities equivalent to (1.9) and (1.10) are obtained from the latter by substituting in them η for ξ and superscript minus for plus.

2. Let us substantiate the statements made above about the applicability of (1.7)—(1.10). We begin with long pipes, and evaluate the terms rejected in (1.6) in the case of a c^--wave, restricting the analysis to the interval $t_1 \leqslant t \leqslant t_2$ in which the wave is intersected by a single c^+-shock. As implied by (1.7), along the c^--characteristic $\eta = \text{const}$ and

$$2dt = d\xi - \varepsilon(\varkappa + 1)(J^- dx + J^+ dx + x dJ^+) / 4$$

Hence, with allowance for (1.7), we have

$$2\int_{t_1}^{t_2} J^+ dt = \left(\int_{\xi_1}^{\xi_{s1}} + \int_{\xi_{s2}}^{\xi_2}\right) J^+ d\xi - \varepsilon\frac{\varkappa + 1}{4}\int_{x_1}^{x_2} J^+ J^- dx -$$

$$\varepsilon\frac{\varkappa + 1}{4}\left[\int_{x_1}^{x_2} (J^+)^2 dx - \left(\int_{J_1^+}^{J_{s1}^+} + \int_{J_{s2}^+}^{J_2^+}\right) J^+ x dJ^+\right] = \int_{\xi_1}^{\xi_2} J^+ d\xi -$$

$$\varepsilon\frac{\varkappa + 1}{8}\left\{\int_{x_1}^{x_2} [2J^+J^- + (J^+)^2]\, dx + (J_2^+)^2 x_2 - (J_1^+)^2 x_1\right\} -$$

$$\int_{\xi_{s1}}^{\xi_{s2}} J^+ d\xi + \varepsilon\frac{\varkappa + 1}{8} x_s [(J_{s2}^+)^2 - (J_{s1}^+)^2] = \int_{\xi_1}^{\xi_2} J^+ d\xi -$$

$$\varepsilon\frac{\varkappa + 1}{8}\left\{\int_{x_1}^{x_2} [2J^+J^- + (J^+)^2]\, dx + (J_2^+)^2 x_2 - (J_1^+)^2 x_1\right\} -$$

$$\int_{\xi_{s1}}^{\xi_{s2}} J^+ d\xi + \frac{1}{2}(J_{s1}^+ + J_{s2}^+)(\xi_{s2} - \xi_{s1})$$

Then, omitting on the strength of (1.9) the last two terms and carrying out summation over all segments of the c^--characteristic, from $t = \eta$ to the current t we obtain

$$2\int_{\eta}^{t} J^+ dt = \int_{\xi(\eta)}^{\xi(t)} J^+ d\xi - \varepsilon\frac{\varkappa + 1}{8}\left\{\int_{X}^{x} [2J^+J^- + (J^+)^2]\, dx +\right.$$

$$(J^+[\xi(t)])^2x - (J^+[\xi(\eta)])^2X\} = O(T) + O(\varepsilon X)$$

which shows that the respective terms $O(\varepsilon T + \varepsilon^2 X)$ in (1.6) are small in comparison with εX even when $n \gg 1$. We stress that the mutual cancellation of the last two terms in the previous equality on the strength of (1.9) is of fundamental importance for long pipes. Indeed it can be shown that each of these terms is $O(\varepsilon X)$ and, consequently, summation over all segments of the c^--characteristic would yield $O(\varepsilon X^2/T)$ and could not be neglected. A similar analysis for the c^--characteristics and shock waves yields the same result.

The nonlinear effects are particularly important in near-resonance modes in the case of short pipes $(n \sim 1)$. It can be shown that in the problems considered below it is necessary to know exactly the relative, not the absolute shifts of intersecting characteristic and shocks of the same set and, also, the instants at which the c^--characteristics and shock waves reach the cross section $x = 0$. For this (1.7)—(1.10) are suitable for taking the nonlinearity into account with an accuracy to within ε, although such accuracy is not always sufficient.

Let us first see how waves of the opposite set affect the inter-section of characteristics and shock of the same set and, consequently, their evolution. Along the pipe length characteristics and shocks of the same intersect each other. They belong to "short waves" whose width with respect to x or t is of order $\varepsilon X \sim \varepsilon T \ll T$. Perturbations of the opposite set shift each short wave as a whole with an accuracy to higher order terms, without affecting the interaction between its "elements". From this point of view, the approximation of noninteracting waves of different sets defines very accurately the flow. As regards the accuracy of calculation of the instant at which the c^--characteristic reach cross section $x = 0$, it also proves to be higher than for intermediate cross sections.

Thus, as shown below, the remainder $t - \xi$, where ξ is the instant of emergence from the left-hand end of the pipe that of the c^+-characteristics which, after having become the c^--characteristic on reflection at $x = X$, returns to the left-hand end at instant t, is important for the analysis in the neighborhood of the "half-wave" resonance when $2n = k + 1 + \Delta$, $k = 0, 1, \ldots$ and $|\Delta| \ll 1$. Close to the "quarter-wave" resonance, when $4n = 2k + 1 + \Delta$, the same part is played by the remainder $t - \zeta = (t - \xi) + (\xi - \zeta)$ in which the instants t, ξ and t, ζ are related to each other as in the previous case. If γ and γ_1 are the contributions of interaction with waves of the opposite set to the remainders $t - \xi$ and $t - \zeta$, it is possible to show that γ and $\gamma_1 = O(\varepsilon n + \Delta)$ hold for the half- and quarter-wave resonances. By virtue of (1.4) the estimates confirm the validity of above statements.

3. At the right-hand end $(x = X)$ of the pipe we stipulate the condition $u(t, X) = 0$ or

$$J^-(t, X) = -J^+(t, X) \tag{3.1}$$

By virtue of (1.6) and (3.1) we have

$$\begin{aligned} J^-[t(\xi), 0] &= -J^+(\xi, 0) + O(\varepsilon^2 n) \\ t(\xi) &= \xi + 2X - \varepsilon X(\varkappa + 1)J^+(\xi, 0)/2 + O(\varepsilon^2 X + \varepsilon\gamma T) \end{aligned} \tag{3.2}$$

where it is taken into account that the number of shocks is $N \sim n$. At the left-hand end we stipulate the boundary condition

$$\varepsilon\alpha[J^+(t, 0) + \beta J^-(t, 0)] = f(\tau) \equiv \delta F(\tau), \quad \tau = t/T \tag{3.3}$$

where α and β are known constants, f and F are periodic functions of τ of period unity, and $\delta = \max|f|$. If velocity oscillates for $x = 0$ then $\alpha = 1/2$ and $\beta = 1$. For pressure oscillations conforming to the law $p = p_0[1 + f(\tau)]$ we have $\alpha = \varkappa/2$ and $\beta = -1$. When in the problem of velocity oscillations at the right-hand pipe end is fixed instead of velocity, then $J^-(t, X) = J^+(t, X)$ holds instead of (3.1) and $\alpha = 1/2$ but $\beta = -1$. Such problem was considered, for instance, in /6--8,10/. Its solution is virtually the same as that of the pressure oscillation problem.

Setting $J(\tau) = J^+(t, 0)$ and $\xi^\circ = \xi/T$ we find that on the strength of (3.2) and (3.3)

$$\begin{aligned} \varepsilon\alpha[J(\tau) - \beta J(\xi^\circ)] &= \delta F(\tau) + O(\varepsilon^3 n) \\ \tau &= \xi^\circ + 2n - (\varepsilon n/2)(\varkappa + 1)J(\xi^\circ) + O(\varepsilon^2 n + \varepsilon\gamma) \end{aligned} \tag{3.4}$$

The same or nearly the same system was previously obtained in /6,7/. However the derivation of discontinuous solutions requires the additional condition of "multivalence elimination" which is to be based on respective equations or laws, such as, for example, (1.9). Unlike in the cited above publications, these rules are incorporated in the procedure itself of the solution derivation in every period of τ proposed here. This not only eliminates the possibility of indefiniteness but, also, makes possible a natural derivation of solutions with any arbitrary number of shocks on a single period. The determination of $J(\tau)$ comprised the establishment of periodicity in τ, and was carried out as follows. First, $J = 0$ was specified for $\tau \leqslant 0$. Then, using (3.4) J was determined for $0 < \tau \leqslant 1$ with simultaneous elimination of ambiguity in conformity with rule (1.9). Owing to the periodicity of the sought solution, the obtained $J(\tau)$ was periodically continued to the negative τ required for the determination of

J in the interval $1 < \tau \leqslant 2$. After the determination of J the process was repeated in every new period. Note that when $F(\tau)$ and β are fixed then, by virtue of equations that determine the solution, $\varepsilon\alpha J/\delta$ is a function of τ and "similarity parameters" $\Omega \equiv \delta n(\varkappa + 1)/\alpha$ and n, more strictly τ, Ω and $\nu \equiv 2n - [2n]$, where $[\varphi]$ is the integral part of φ. Before presenting the results of calculations, let us consider the case when system (3.4) can be simplified. If $\varepsilon^2 \ll \delta$ then, retaining in the first of Eqs. (3.4) only the terms that are linear with respect to ε, we obtain the equations

$$\varepsilon\alpha\,[J(\tau) - \beta J(\tau - 2n)] = \delta F(\tau) \tag{3.5}$$

which defines the oscillations of gas in the usual acoustic approximation. Let, moreover, $F(\tau) = \sin 2\pi\tau$. Since any periodic function with zero mean value can be expanded in a Fourier series in sines, it is possible with the use of the latter to derive a solution in the general case. For function $F(\tau)$ the periodic solution of (3.5) is

$$\frac{\varepsilon\alpha J}{\delta} = \frac{\sin(2\pi\tau + \varphi)}{\sqrt{1 + \beta^2 - 2\beta\cos 4\pi n}}, \quad \operatorname{tg}\varphi = \frac{\beta\sin 4\pi n}{\beta\cos 4\pi n - 1} \tag{3.6}$$

In the presence of resonance $1 + \beta^2 - 2\beta\cos 4\pi n = 0$ and (3.6) loses its meaning. Since $|\cos 4\pi n| \leqslant 1$ and $1 + \beta^2 \geqslant 2|\beta|$ in which the equality is only satisfied for $|\beta| = 1$, a resonance can only occur when $\beta = \pm 1$. In that case the conditions of resonance reduce to the equalities $1 \mp \cos 4\pi n = 0$, which yields $2n = k + 1$ and $4n = 2k + 1$ $(k = 0, 1, \ldots)$, respectively. At near-resonance (when $|\Delta| \lesssim \delta$) by virtue of (3.6) $\varepsilon \geqslant O(1)$, i.e. the assumption that $\varepsilon^2 n \ll \delta$ used in the derivation of (3.5) is violated. In connection with the investigation of resonance it is useful to write down the zero solution of (3.5) for $F(\tau) = \theta(\tau)\sin 2\pi\tau$ with $\tau \leqslant 0$, and the Heaviside function $\theta(\tau)$ equal zero for $\tau < 0$ and unity for $\tau \geqslant 0$. It can be shown that for $\tau \geqslant 0$ it is of the form

$$\frac{\varepsilon\alpha}{\delta}J = \frac{\tau\sin 2\pi\tau}{K} + \frac{1+\beta}{4\pi K} + \frac{\sin 2\pi\tau}{2} + \frac{\cos 2\pi\tau}{4\pi K} + \frac{K}{\pi}\sum_m \frac{\cos(2\pi M\tau/K)}{K^2 - M^2}$$

$$K(k, \beta) = k + \frac{3+\beta}{4}, \quad M(m, \beta) = m + \frac{\beta - 1}{4}$$

where summation is carried out over all $m = 0, 1, \ldots$, except $m = k + 1$.

Another simplified form of (3.4) valid when $\varepsilon n \ll 1$ is obtained as follows. Expanding $J(\xi^\circ)$ in the neighborhood of $\xi^\circ = \tau - 2n$ we obtain in conformity with the second of equalities (3.4) the equation

$$\begin{aligned}\varepsilon\alpha\,[J(\tau) - \beta J(\tau - 2n) - (\varepsilon\beta n/2)(\varkappa + 1)J(\tau - 2n)J^\cdot(\tau - 2n)] ={}& \\ \delta F(\tau) + O(\varepsilon^3 n + \varepsilon^2\gamma)&\end{aligned} \tag{3.7}$$

derived on the assumption that $J^\cdot \equiv dJ/d\tau \sim O(1)$. This estimate is not valid, for instance, for a beam of rarefaction waves where $J^\cdot \sim \varepsilon^{-1}$ and, also, for uneven functions $F(\tau)$. Note that passing to (3.7) and (3.8) in small neighborhoods (of order ε or Δ) of shocks introduces additional errors.

At near-resonance modes in the problem on velocity oscillations $2n = k + 1 + \Delta$, and γ is close to the zero integral of J over the $(k + 1)$-st period and, as previously indicated, is $O(\varepsilon n + \Delta)$. Hence, having expanded $J(\tau - 2n)$ in (3.7) in the neighborhood of point $\tau - k - 1$ and taking into account the periodicity of J, we obtain the equation

$$\begin{aligned}\varepsilon\,[J^\cdot(\tau)\Delta - (\varepsilon n/2)(\varkappa + 1)J(\tau)J^\cdot(\tau)] = 2\delta F(\tau) + O(\varepsilon^3 n^2 +{}& \\ \varepsilon^2\Delta + \varepsilon\Delta^2)&\end{aligned} \tag{3.8}$$

which is valid, as is also Eq. (3.17) in /1/, only for $\varepsilon n \ll 1$, or more exactly for $\varepsilon nJ^\cdot \ll 1$. The essential difference in these is the substitution in /1/ of $\pi^{-1}\operatorname{tg}(\pi\Delta)$ for Δ which makes the equation in /1/ and (3.8) valid close to and away from resonance. Without this substitution (3.8) is only valid close to resonance where $|\Delta| \lesssim O(\delta)$, and by virtue of (3.8) $\varepsilon \sim \delta^{1/2}$. Unlike (3.7), (3.8), and Eq. (3.17) in /1/, system (3.4) together with rule (1.9) is valid independently of the quantity $\varepsilon nJ^\cdot$ or of the degree or nearness to resonance.

In the problem of pressure oscillations $\alpha = \varkappa/2$ and $\beta = -1$, and the near-resonance modes obtain for $4n = 2k + 1 + \Delta$ with $|\Delta| \lesssim \delta \ll 1$. To analyze these we write (3.4) for two instants of time τ and ξ° with the relation of ζ° to ξ° being the same as that of ξ° to τ. We have

$$\begin{aligned}
&J(\tau) + J(\xi^\circ) = (2\delta/\varepsilon\varkappa)F(\tau) + O(\varepsilon^2 n)\\
&J(\xi^\circ) + J(\zeta^\circ) = (2\delta/\varepsilon\varkappa)F(\xi^\circ) + O(\varepsilon^2 n)\\
&\xi^\circ = \tau - 2n + (\varepsilon n/2)(\varkappa + 1)J(\xi^\circ) + O(\varepsilon^2 n + \varepsilon\gamma)\\
&\zeta^\circ = \xi^\circ - 2n + (\varepsilon n/2)(\varkappa + 1)J(\zeta^\circ) + O(\varepsilon^2 n + \varepsilon\gamma)
\end{aligned} \tag{3.9}$$

We add the third and fourth of equalities (3.9), eliminate from the right-hand side of the obtained equation the sum $J(\xi^\circ) + J(\zeta^\circ)$ using the second equality of that system, and take into account the structure of terms in (1.4), which provided in (3.4) and (3.9) terms of order $\varepsilon\gamma$. Then setting $4n = 2k + 1 + \Delta$ we obtain

$$\zeta^\circ = \tau - 2k - 1 - \Delta + (\delta n/\varkappa)(\varkappa + 1)F(\xi^\circ) + O(\varepsilon^3 n^2 + \varepsilon^2 n + \varepsilon\gamma_1)$$

where in conformity with the earlier statement $\gamma_1 = O(\varepsilon n + \Delta)$ when $|\Delta| \ll 1$. Restricting the analysis to $|\Delta| \ll 1$ we subtract the second of Eqs. (3.9) from the first and use for ζ° the last expression. Then, proceeding as in the derivation of (3.7) and (3.8), we obtain the equation

$$\varepsilon \Delta \varkappa J^{\cdot}(\tau) - \varepsilon n \delta (\varkappa + 1) F(\xi^\circ) J^{\cdot}(\tau) = 2\delta [F(\tau) - F(\xi^\circ)] + O(\varepsilon^3 n + \varepsilon^4 n^2 + \varepsilon^2 \Delta + \varepsilon \delta^2 n^2) \tag{3.10}$$

In this problem (see Sect.2), unlike in the previous one, $\gamma \sim O(1)$. Hence in this case Eq. (3.10) is more exact then (3.4). This is readily understood, if one recalls that in the derivation of (3.10) supplementary information about terms related to wave interaction was used besides (3.4).

We begin the investigation of near-resonance modes on long pipes ($n \gg 1$) for which we can expect that $\varepsilon n \delta = \chi \delta \gg \max(|\Delta|\varepsilon, \varepsilon^3 n, \varepsilon^4 n^2, \varepsilon \delta^2 n^2)$ with $\chi \sim O(1)$, using (3.10). In this case $\varepsilon = \chi / n$ is independent of δ and (3.10) reduces to

$$\varepsilon n J^{\cdot}(\tau) = 2[F(\tau - 1/2) - F(\tau)] / (\varkappa + 1) F(\tau - 1/2) \tag{3.11}$$

In conformity with the equalities and inequalities obtained above and, also, with the meaning of ε and δ the necessary condition of realizing the considered approximation are: $\chi / n \gg \delta \gg \max(\chi |\Delta| n^{-1}, \chi^3 n^{-2})$, but they are insufficient. Indeed, as implied by the derivation of (3.10) and (3.11), the continuity sections of $J(\tau)$ are defined by these equations only then, when they are the result of double reflection from the $x = X$ wall of characteristics generated by the continuous distribution of $J(\zeta^\circ)$. If F is a function odd with respect to the half-period as, for example, $\sin 2\pi\tau$, then $F(\tau - 1/2) = -F(\tau)$ and $\varepsilon n J^{\cdot}(\tau) = 4/(\varkappa + 1)$. On the other hand according to (3.9)

$$\xi^\circ = \tau - 2n + \varepsilon n J(\xi^\circ)(\varkappa + 1)/2$$

and an analogous relation links ζ° and ξ°. It follows from this that the continuous solution (3.11) is completely "upset", and becomes a shock when passing the pipe in one direction. Hence in the case of long pipes function $J(\tau)$ does not contain such sections, at least in the case of odd F. Consequently, the section $x = 0$ can only be reached, besides shock waves, by the c-characteristics that are reflections from the $x = X$ cross section of centered waves originating at the reflection of compression shocks from the pipe left-hand end. This is diagrammatically shown in Fig.1, where the thin lines represent characteristics and the heavy ones, shocks.

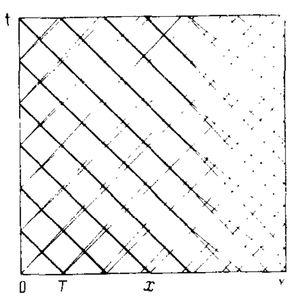

Fig.1

Of particular interest is the case in which the centered wave width in the direction of t is, after return to section $x = 0$, equal T. It can be shown that in the case of long pipes this occurs at resonance. The initial intensity with respect to εJ of each beam is, then, equal $1/(\varkappa + 1)n$ and the instants of shock reflection from section $x = 0$ coincide for $F = \sin 2\pi\tau$ with the half-periods. Although in this case (3.10) and (3.11) are invalid, nevertheless $\varepsilon n \sim O(1)$ which by virtue of (3.9) ensures the validity of the derived solution, in any case for $n^{-1} \sim \delta \gg n^{-2}$. Such solutions can also be constructed using (3.9) also for $n \leqslant O(1)$. Here $\varepsilon \sim O(1)$ and the rarefaction wave fan is the result of reflection from cross section $x = 0$ of a beam of compression waves, not of a shock wave (in fact, compression waves containing the shock may focus not on a point but on a small segment of the t-axis). These solutions are, however, of no interest, since for them the terms under the symbol "O" in (3.9) are not small. In the case of pressure oscillations at one of the short pipe ends the analysis of near-resonance modes cannot generally be carried out in the considered approximation as completely as in the investigated above cases.

The following possibilities exist here besides the one just now discarded.

Let $|\Delta| \ll 1$ but still sufficiently large for $\varepsilon |\Delta| \sim \delta \gg \max(\varepsilon^3, \varepsilon^2 |\Delta|)$. Then $\varepsilon \sim \delta / \Delta$, and the corresponding simplification of equality (3.10) coincides with the equation that can be obtained from Eq. (3.5) of acoustics when $|\Delta| \ll 1$. The latter is natural, if one takes into account that in this case the inequality $|\Delta| \gg \delta$, which ensures remoteness from resonance, follows from $\delta \gg \max(\varepsilon^3, \varepsilon^2 |\Delta|)$.

Let us, finally, consider a short pipe for $|\Delta| \leqslant \delta$, when in (3.10) the terms of order $O(\varepsilon^3)$ can be the principal term besides the first one. From this, even without knowing its explicit expression, we find that $\varepsilon \sim \delta^{1/3}$, as opposed to $\varepsilon \sim \delta^{1/2}$ in the problem of velocity variation. Note that the necessity of including in the analysis of the problem of pressure oscillations or, which is the same, of problems of the pipe with fixed pressure at one of its ends, follows from the investigations described in /6,8—10/, with the dependence $\varepsilon \sim \delta^{1/3}$ first established in /10/ and then in /8/.

Because of the importance of term with third power of ε for near-resonance modes with specified and oscillating pressure, we recall the factors that induce their appearance. This is, first of all, the transition from (1.4) to (1.6). If this is not carried out, the second of Eqs. (3.4) is transformed to the more exact equality

$$\tau = \xi^\circ + [2n + O(\varepsilon\gamma)] / [1 + \varepsilon(\varkappa + 1) J(\xi^\circ) / 4]$$

The exactness of the obtained solution is, nevertheless, even now insufficient due to a number of reasons. The main of these are: the disregard of interaction between waves of different sets and the change of invariants at the shocks (the inexpediency of taking into account the entropy increase was explained at the beginning of this paper), and the use of rule (1.9). As regards the linearization of the boundary condition for $x = 0$ in the problem of velocity variation it can be stated that the errors associated with this although of the same order, are

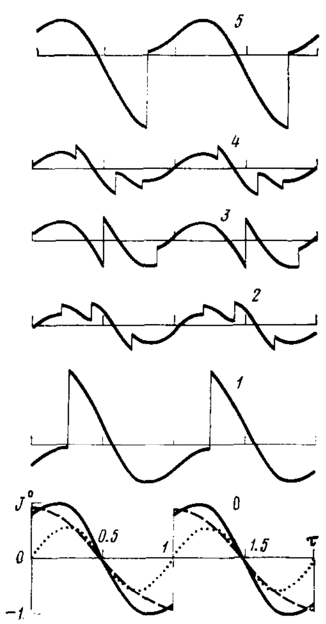

Fig.2

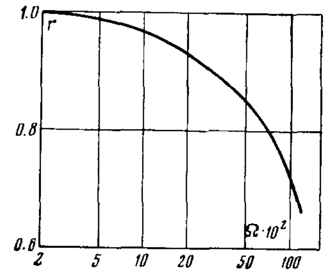

Fig.3

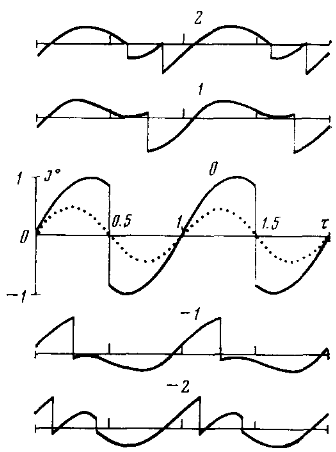

Fig.4

not of fundamental importance, since the condition $\varepsilon u(t, 0) = f(\tau)$ may be considered as an exact equality and unrelated to the linearization of the boundary condition in the problem of the piston.

4. As already indicated, a numerical procedure was developed during the investigation, which made possible an effective solution of obtained equations, of course, without taking into account terms of order $\varepsilon^2 n$, etc. Its effectiveness was confirmed by numerous computations a small part of which was carried out for $F(\tau) = \sin 2\pi\tau$ and $\varkappa = 1.4$ is shown in Figs.2—4.

The results of calculations in the problem of velocity oscillations were compared with respective results obtained in /1/. As already indicated, the formulas used there are valid only for comparatively small $K_\varepsilon \equiv \varepsilon n \max |J'(\tau)|$. In the considered examples the factor $\max |J'(\tau)| \approx 2\pi$ and is consequently important. The results obtained with $K_\varepsilon \ll 1$ by various methods were virtually the same. This and equality (1.10) which implies that the integral over the period of $J(\tau)$ must remain invariant which was always checked, may be taken as the justification of $K_\varepsilon \ll 1$ taken in /1/ as the rule for the introduction of discontinuity. As K_ε or $K_\delta \equiv 2\pi\delta n$ is increased, the difference in results becomes not only quantitative but, also, qualitative. This is demonstrated in Fig.2 where the solid lines relate to six "oscillograms": $J^\circ(\tau) \equiv \varepsilon J(\tau) / (\varepsilon J)^*$, where at resonance $(\varepsilon J)^* = \max(\varepsilon J)$. In this case the velocity oscillation amplitude at $x = 0$ was fairly high: $\delta(\varkappa + 1) = 0.08$ and $n \approx 3$, hence $K_\delta \sim O(1)$. The resonance oscillogram denoted by the symbol "0" in Fig.2 relates to $2n = 6, k = 5$, and $\Delta = \nu \equiv 2n - [2n] = 0$, where the integer k is defined in the equality $2n = k + 1 + \Delta$. The solid curves denoted in Fig. 2 by numerals $1, \ldots, 5$ relate to the following sets of values of $2n, k, \nu$, and Δ: 6.15, 5, 0.15, 0.15; 6.4, 5, 0.4, 0.4; 6.5, 5(6), 0.5, 0.5(−0.5); 6.6, 5(6), 0.6. 0.6(−0.4); 6.85, 5(6), 0.85, 0.85(−0.15). In the last three sets the figures in parentheses indicate the "supplementary values of k and Δ which define the closeness of the mode to the adjacent resonance". In this problem, as implied by (3.4) and (1.9), we have

$$J(\tau, \Omega, -\Delta) = -J(-\tau, \Omega, \Delta) \tag{4.1}$$

The similarity parameter $\Omega \equiv \delta n(\varkappa + 1)/\alpha$ differs not more that 15% even for the outer curves in Fig.2. The comparison of curves 1 and 5, or 2 and 4 indicates the validity of (4.1). The sinusoid $\delta F(\tau)/(\varepsilon J)^*$ shown by the dotted curve in Fig.2 defines velocity oscillations at $x = 0$, while the dash line is the oscillogram calculated in conformity with /1/. The difference between the solid and dash line curves shows the error of the theory in /1/ (when $K_\delta \sim O(1)$). Note that for resonances in which more than one shock is formed in a period, the rule of shock introduction used in /1/ indicates a single discontinuity.

For $K_\varepsilon \ll 1$ on the strength of (3.4) and (3.8), as well as of formulas in /1/ close to a "half-wave" resonance $\varepsilon \sim \delta^{1/2}$, or in "similarity variables" $\varepsilon / \delta \sim \Omega^{-1/2}$. In particular, the difference of εJ at the shock proves to be equal $4\delta/\sqrt{\pi\Omega}$. As Ω and, consequently, also K_δ and K_ε, are increased, deviation from this regularity is observed. This is illustrated in Fig.3 where the ratio of the shock intensity r to $4\delta/\sqrt{\pi\Omega}$ is shown as a function of Ω.

The oscillograms shown in Fig.4 were computed on the same basis as in Fig.2 and relate to the problem of pressure oscillation with $\delta(\varkappa + 1)/\varkappa = 0.08$ and $k = 6$. In this case $4n = 2k + 1 + \Delta$ corresponds to the "quarter-wave" resonance $(\Delta = 0)$; it is shown by the curve denoted by 0, while the curves denoted by numerals $1, -1, 2$, and -2 relate to the following "sets" of $2n, \nu$ and Δ: 6.65, 0.65, 0.3; 6.35, 0.35, −0.3, 6.75, 0.75, 0.5 6.25, 0.25, and −0.5. As in the previous case, equality (4.1) is satisfied, as can be seen from Fig.4. In Fig.2 the oscillogram discontinuities correspond to compression shocks propagating to the right from the pipe left-hand end, while those in Fig.4 relate to rarefaction wave beams. Both are the result of reflection of waves arriving here from the right.

The authors thank A. N. Rudenko for bringing this problem to their attention.

REFERENCES

1. CHESTER, W., Resonant oscillations in closed tubes. J. Fluid Mech., Vol.18, pt.1, 1964.
2. LIN, C. C., On a perturbation theory based on the method of characteristics. J. Math. and Phys., Vol.33, No.2, 1954.
3. BOA-TEH CHU and TING, S. J., Thermally driven nonlinear oscillations in a pipe with travelling shock waves. Phys. Fluids, Vol.6, No.11, 1963.
4. BOA-TEH CHU, Analysis of a self-sustained thermally driven nonlinear vibration. Phys.Fluids, Vol.6, No.11, 1963.
5. MORTELL, M.P. and VARLEY, E., Finite amplitude waves in bounded media: nonlinear free vibrations of an elastic panel. Proc. Roy. Soc. Ser. A, Vol.318, No.1533, 1970.
6. MORTELL, M. P., Resonant oscillations: a regular perturbations approach. J. Math. Phys., Vol.12, No.7, 1971.
7. SEYMOUR, B. R. and MORTELL, M. P., Resonant acoustic oscillations with damping: small rate theory. J. Fluid Mech. Vol.58, pt.2, 1973.
8. JIMENEZ, J., Non-linear gas oscillations in pipes, pt. 1. Theory. J. Fluid.Mech., Vol.59, pt.1, 1973.
9. KELLER, J. J., Third order resonances in closed tubes. ZAMP, Vol.27, No.3, 1976.
10. COLLINS, W. D., Forced oscillations of systems governed by one-dimensional non-linear wave equations. Quart. J. Mech. and Appl. Math., Vol.24, p.2, 1971.
11. CHESTER, W., Resonant oscillations of water waves. I. Theory. Proc. Roy. Soc., Ser. A, Vol. 306, No. 1484, 1968.
12. KELLER, J. J., Subharmonic non-linear acoustic resonances in closed tubes. ZAMP, Vol.26, No.4, 1975.
13. KELLER, J. J., Resonant oscillations in closed tubes: the solution of the Chester equation. J. Fluid. Mech., Vol.72, pt.2, 1976.

14. KELLER, J. J., Nonlinear acoustic resonances in shock tubes with varying cross sectional area. ZAMP, Vol.28, No.1, 1977.

15. LANDAU, L. D., On shock waves at large distances from the place of their generation,PMM, Vol.9, No.4, 1945.

16. LANDAU, L. D. and LIFSHITS, E. M., Mechanics of Continuous Media. English translation, Pergamon Press.

17. WHITHAM, G. B., Linear and Nonlinear Waves. Moscow, "Mir", 1977.

18. SIMONS, G. A., Decay of a diamond shock pattern. AIAA Journal, Vol.10, No.8, 1972.

19. KRAIKO, A. N. and OSIPOV, A. A., The damping of periodic sequence of weak shock waves in channels with sound-absorbing walls. Izv. Akad Nauk, SSSR, MZhG, No.4, 1976.

20. DUMNOV, G. E., Oscillations of gas in pipes subjected to periodically varying pressure. Izv. Akad.Nauk SSSR, MZhG, No.5, 1978.

Translated by J.J.D.